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# GAME PROBLEM OF IMPULSE "HARD" CONTACT IN A POSITION ATTRACTION FIELD WITH AN OPPONENT WHO REALIZES A BOUNDED THRUST

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We examine a game problem [1, 2] of the "hard" contact of two points (players) in a linear position field of attraction to a fixed center. We assume that the first (the minimizing) player realizes a control (thrust, force) which is bounded in total momentum, while the second (the maximizing) player has available a controlled thrust which is bounded in absolute value. The game's value is the time up to "hard" contact, i.e. the geometric coincidence of the points for an arbitrary relative velocity. This paper abuts [3] in subject matter and is very closely related to [4]. In those sections which repeat the material of [4] the proofs are given concisely.

**1.** Let two points (the first and second players) with masses $m_1$, $m_2$ be attracted to a fixed center $O$ by forces $F_{1,2} = -\omega^2 m_{1,2} r_{1,2}$, where $\omega^2 = \text{const} > 0$, while $r_{1,2}$ are the position radius vectors of the points relative to center $O$. Suppose that the players have available arbitrarily-directed control forces (thrusts) $f_{1,2}$ with the constraint $|f_2| \leqslant \nu = \text{const} > 0$ for the second player. By a selection of scales we can obtain $\omega^2 = 1$, $\nu = m_2$. Suppose that after this the constraint on $f_1 = m_1 u$ – the first player's thrust – takes the form of the momentum constraint

$$\mu^{(1)} = \mu_0 - \int_0^\tau |u|\, dt \geqslant 0 \tag{1.1}$$

If force $f_1$ is finite, then in the variables

$$x_1 = r_1 - r_2, \quad y_1 = r_1^{\cdot} - r_2^{\cdot}, \quad f_2 = -m_2 v$$

the motion develops according to the equations

$$\begin{gathered} x_1^{\cdot} = y_1, \qquad y_1^{\cdot} = -x_1 + u + v \\ \mu^{\cdot} = \mu_1^{(1)\cdot} = -|u|, \quad \mu \geqslant 0, \quad |v| \leqslant 1 \end{gathered} \tag{1.2}$$

In the region $x = |x_1| > 0$, using the notation

$$j_\alpha = x_1 / x, \quad y_\alpha = (y_1^{\cdot} j_\alpha), \quad y_{\beta,1} = y_1 - y_\alpha j_\alpha$$

$$y_\beta = |y_{\beta,1}|, \quad j_\beta = y_{\beta,1} / y_\beta, \quad j_\gamma \perp j_\alpha, \quad j_\alpha \perp j_\beta, \quad y_\beta > 0$$

in the moving system of unit vectors $j_\alpha$, $j_\beta$, $j_\gamma$ we obtain, as a consequence of Eqs. (1.2),

$$x^{\cdot} = y_\alpha, \quad y_\alpha^{\cdot} = -x + u_\alpha + v_\alpha + y_\beta^2 / x$$

$$y_\beta^{\cdot} = u_\beta + v_\beta - y_\alpha y_\beta / x, \quad \mu^{\cdot} = -|u|, \quad \mu \geqslant 0, \quad |v| \leqslant 1$$

For $y_\beta = 0$ the unit vectors $j_\beta$, $j_\gamma$ are arbitrary in a plane normal to $j_\alpha$. In this case

$$y_\beta^{\cdot} = \sqrt{(u_\beta + v_\beta)^2 + (u_\gamma + v_\gamma)^2}$$

Constraint (1.1) admits of jumps in the position $w\ (x,\ y_\alpha,\ y_\beta,\ \mu)$ to the values

$$w^{(1)}\,[x, y_\alpha^{(1)} = y_\alpha + \mu_{1,\alpha},\quad y_\beta^{(1)} = \sqrt{(y_\beta + \mu_{1,\beta})^2 + \mu_{1,\gamma}^2},\quad \mu^{(1)} = \mu - |\mu_1| \geqslant 0]$$

where the "impulse" control $u = \mu_1\delta$ has been used. The vector $w^{(1)}$ is the result of the impulse.

The definition of the admissible pairs $u\ (w,\ v)$, $v\ (w)$ and of the trajectories

$$w^{(1)}\ (w\ (t = 0),\quad \{u\ (w,\ v),\ v\ (w)\},\ t) = w^{(1)}\ (\cdot, t)$$

corresponding to them repeats the corresponding definitions in [4].

Let us consider two closed sets

$$M_1\ [x = 0],\quad M_2\ [\mu - \pi / 2 - |\ y_1\ | \equiv \mu - \pi / 2 - y = 0]$$

in the position space $W$. Set $M_1$ is the set of "hard" contact, while set $M_2$ plays an auxiliary role. Denoting by $t_j\ (w\ (\cdot)) = T_j\ [u,\ v]$ the instants at which the positions first hit onto set $M_j$, we call the pair $u_j^\circ$, $v_j^\circ$ and the time $T_j^\circ = T_j\ [u_j^\circ,\ v_j^\circ]$ optimal ones when the estimates

$$T_j\ [u_j^\circ,\ v] \leqslant T_j^\circ \leqslant T_j\ [u,\ v_j^\circ]$$

are satisfied. The control $v_{0,j}$ and the region $W_{0,j} \notin M_j$ are called the control and the evasion region if the inclusion $w^{(1)}\ (\cdot) \in W_{0,j}$ is preserved for all $t \geqslant 0$ by any pair $u$, $v_{0,j}$.

2. By analogy with [4] we consider the functions

$$q\,(w, p) = \mu - \sqrt{y_\beta^2 + (p - y_\alpha)^2} - \operatorname{arc\,tg} p / x - \pi / 2$$
$$r_1\ (w) = \max_p q\ (w,\ p \leqslant 0)$$

where $p_1\ (w) \leqslant 0$ is the point where the maximum of $r_1\ (w)$ is realized, $p_2\ (w)$ is the smallest zero of the function $q\ (w,\ p)$, existing and nonpositive in the region $C_1\ [r_1\ (w) \geqslant 0]$. In the notation

$$l_1\ (w,\ p) = \sqrt{x^2 + p^2},\quad l_2\ (w,\ p) = \sqrt{y_\beta^2 + (p - y_\alpha)^2}$$

we obtain, in the region $w \notin M_1 \bigcup\ [y_\beta = 0]$, the equalities

$$q_p' = -(p - y_\alpha) / l_2\ (w,\ p) - x / l_1^2\ (w,\ p)$$
$$q_p'' = -y_\beta^2 / l_2^3\ (w,\ p) + 2px / l_1^4\ (w,\ p)$$

The form of the second derivative and the equality $\lim q_p' = +1$ as $p \to -\infty$ show that the function $q\ (w,\ p)$ can have the only possible stationary maximum in the region $p \leqslant 0$, which undoubtedly exists in the region $w \in [x > 0,\ y_\beta > 0,\ q_p'\ (w,\ 0) \leqslant 0]$. The derivative $q_p'$ may have discontinuities when $y_\beta = 0$. Omitting the elementary details, we cite the result

$$p_1\,(w) = -\sqrt{x - x^2},\quad w \in [y_\beta = 0,\quad 0 < x \leqslant 1, \sqrt{x - x^2} + y_\alpha \geqslant 0]$$

$$p_1(w) = y_\alpha, \quad w \in [y_\beta = 0] \cap \{[x > 1, \quad y_\alpha \leqslant 0] \cap [0 < x \leqslant 1$$
$$\sqrt{x - x^2} + y_\alpha < 0]\}$$

All the regions mentioned satisfy the condition $q_p'(w, \ p \to +0) \leqslant 0$. We denote their union by $D_1$. In the remaining part $D_2 = W \setminus (M_1 \cup D_1)$ of the space the function $q(w, p)$ grows monotonically for $p < 0$ and $p_1(w) = 0$.

Acting by the scheme in [4], we can prove the validity of the following statements.

2.1. The functions $r_1 (w \notin M_1)$, $p_1(w \notin M_1)$, $p_2 (w \in C_1 [r_1 \geqslant 0])$ are continuous,

2.2. If $\tau$ is the first instant of realization of the equality $\lim\ x(t \to \tau + 0) = 0$, then $\lim\ r_1 (w(\cdot, \ t \to \tau)) \geqslant 0$.

2.3. Any impulse control from the family

$$u(w, \ p, \ m) = m\delta\,[(p - y_\alpha)\, j_\alpha - y_\beta j_\beta] \,/\, l_2(w, \ p)$$
$$0 \leqslant m \leqslant \min\,(1, \ \mu \,/\, l_2(w, \ p))$$

preserves the value of $q(w, \ p)$, i.e.

$$\Delta q = q(w^{(1)}, \ p) - q(w, \ p) = 0$$

while any impulse control not occurring in this family realizes the bound $\Delta q < 0$.

2.4. In the region $w \in C_1$ the family of impulse controls

$$mu_1^\circ(w) = m\delta\,[(p_2 - y_\alpha)\, j_\alpha - y_\beta j_\beta] \,/\, l_2(w, \ p_2), \quad 0 \leqslant m \leqslant 1$$

realizes the equality $\Delta p_2 = p_2(w^{(1)}) - p_2(w) = 0$, while any control $u = \mu_1 \delta$ not occurring in this family either strictly increases the root $p_2\ (\Delta p_2 > 0)$ or translates the position into the region $w^{(1)} \in C_2 \equiv W \setminus (C_1 \cup M_1)$.

The derivative $T_1^\cdot$ of the function (for finite $u$)

$$T_1(w) = \operatorname{arc\,tg} p_2 \,/\, x + \pi/2$$

has the form

$$T_1^\cdot = (q_p'(w, \ p_2)\, l_1^2)^{-1}\, [R_1 + R_2 + R_3], \quad w \in C_1 \cap [l_2 > 0]$$
$$R_1 = a_1 l_1^2 \,/\, l_2 - p_2 l_2$$
$$R_2 = x l_2^{-1}\,(|\, u\,|\, l_2 - a_1 u_\alpha + y_\beta u_\beta), \quad R_3 = x l_2^{-1}\,[-a_1 v_\alpha + y_\beta v_\beta]$$
$$T_1^\cdot = -|\, u\,| + s\,(u_\alpha + v_\alpha) + \{(1 - s^2)\,[(u_\beta + v_\beta)^2 + (u_\gamma + v_\gamma)^2]\}^{1/2}$$
$$w \in C_1 \cap [l_2 = 0], \quad s = x \,/\, (x^2 + y_x^2)$$
$$l_1 = \sqrt{x^2 + p_2^2}, \quad l_2 = l_2(w, \ p_2), \quad a_1 = p_2 - y_\alpha$$

These formulas show [4] that the pair

$$u_1^\circ, \quad v_1^\circ = -u_1^\circ \,/\, |\, u_1^\circ\,|, \quad w \in C_1 \cap [l_2 > 0]$$
$$u_1^\circ = -v, \quad v_1^\circ = s j_\alpha + \sqrt{1 - s^2}\, j_\beta, \quad w \in C_1 \cap [l_2 = 0]$$

realizes the estimates

$$T_1^\cdot(w, \ u_1^\circ, \ v) \leqslant T_1^\cdot(w, \ u_1^\circ, \ v_1^0) \leqslant T_1^\cdot(w, \ u, \ v_1^0) \tag{2.1}$$

The first estimate is verified elementarily. Let us demonstrate the validity of the second one. In fact, according to statement 2.4, any impulse control $u = \mu_1\delta$, preserving the inclusion $w^{(1)} \in C_1$ and not equal to $mu_1°$, strictly increases $p_2$ and, consequently, $T_1$. It is not difficulat to verify that the control $mu_1°$ preserves the first summand $a_1 l_1^2 / l_2$ of sum $R_1$ and converts the second summand to the quantity $-p_2(w)(1-m) \times l_2(w, p_2)$. If $p_2(w) < 0$, the control $u_1°$ realizes the minimum of sum $R_1$. If $p_2(w) = 0$, it can be shown that for any $u \neq u_1°$ the estimate $-p_2 l_2 > 0$ is realized at the next instant.

To discuss what is possible to the first player on the boundary $r_1 = 0$ of region $C_1$ we consider the derivative $r_1^{\cdot}$ of the function $r_1(w)$

$$r_1^{\cdot} = 1 + p_1 l_2(w, p_1) + P_2 + P_3, \quad w \in D_1 \cap [l_2(w, p_1) > 0]$$
$$P_2 = -|u| + [(p_1 - y_\alpha) u_\alpha - y_\alpha u_\beta] / l_2(w, p_1)$$
$$P_3 = +[(p_1 - y_\alpha) v_\alpha - y_\beta v_\beta] / l_2(w, p_1)$$
$$r_1^{\cdot} = 1 - |u| - s(u_\alpha + v_\alpha) - \{(1 - s^2)[(u_\beta + v_\beta)^2 + (u_\gamma + v_\gamma)^2]\}^{1/2}$$
$$w \in D_1 \cap [l_2(w, p_1) = 0]$$

If the second player's control in the region $D_1 \cap C_2$ is specified by the formulas

$$v_1 = -[(p_1 - y_\alpha) j_\alpha - y_\beta v_\beta] / l_2(w, p_1)$$
$$w \in D_1 \cap [l_2(w, p_1) > 0]$$
$$v_1 = s j_\alpha + \sqrt{1 - s^2} j_\beta, \quad w \in D_1 \cap [l_2(w, p_1) = 0]$$

and in the region $D_2 \cap C_2$ it is continued uninterruptedly so that it passes continuously into the control $v_1$, $v_1^{\circ}$ on the common boundary of the regions $(D_1 \cap C_2)$ and $(D_2 \cap C_2)$, as well as on the boundary of the regions $(D_2 \cap C_2)$ and $(D_2 \cap C_1)$, then the control $v^1 = v_1^{(0)}$, $w \in C_1$, $v^1 = v_1$, $w \in C_2$ is continuous for $w \in M_1$.

In addition, from the form of the derivative $r_1^{\cdot}$ and from statement 2.3 it follows that any errors of the first player on a part of the boundary $D_1 \cap [r_1 = 0]$ translate the position into region $C_2$. His errors on a part of the boundary $D_2 \cap [r_1 = 0]$ lead to the same result. In fact, according to 2.3 any impulse control not parallel to the optimal one lessens $q(w, 0) = r_1(w)$. Any finite control $u$ which converts the derivative $T_1^{\cdot}$ into a positive quantity also translates the position into region $C_2$.

2.5. In the region $w \in C_2$ the control $v^1$ in pair with any control $u$ effects a contact no earlier than the instant $t = \pi / 2$.

**Proof.** Let $x(\cdot, \tau) = 0$; then we can find $t_1 < \tau$ such that $x(\cdot, t \in [t_1, \tau)) < 1$. From the equality $q_p'(w, 0) = (y_\alpha / y) - (1 / x)$ follow the estimate $q_p'(w(\cdot, t \in [t_1, \tau)), 0) < 0$ and the inclusion $w(\cdot, t \in [t_1, \tau)) \in D_1$. By the construction of control $v^1$ the function $r_1(w)$ does not grow for $w \in D_1 \cap C_2$. This means that $w(\cdot, t \in [t_1, \tau)) \in D_1 \cap C_1$. In fact, from the contrary premise $w(\cdot, t_2 \in [t_1, \tau)) \in D_1 \cap C_2$ and from the nonincrease of $r_1$ follows the estimate $r_1(w(\cdot, t \in [t_2, \tau))) \leqslant r_1(w(\cdot, t_2)) < 0$. This contradicts statement 2.2. From the inclusion $w(\cdot, t_1) \in D_1 \cap C_1$ follows the existence of $t_3$, namely, the first root of the equation $r_1(w(\cdot, t_3)) = 0$, and the function $r_1$ does not decrease along the trajectory when $t = t_3$. It is obvious that this is possible only for

$$p_2(w(\cdot, t_3)) = 0, \quad p_1(w(\cdot, t_3)) \geqslant 0$$

i.e. $w(\cdot, t_3) \in [T_1(w) = \pi / 2] = M_2$, Q.E.D.

The arguments presented above allow us to formulate a result.

Theorem 1. When $w(0) \in C_1$ $[r_1 \geqslant 0]$ the controls $u_1^\circ$, $v_1^\circ$ realize the optimal time $T_1^\circ(w) = \operatorname{arc\,tg} p_2 / x + \pi / 2$.

Proof. Statement 2.1 establishes the continuity of function $T_1^\circ(w)$. Statement 2.4 shows that the first player cannot lessen the function $T_1^\circ(w)$ by an impulse. Estimate (2.1) establishes a saddle point for the derivative when $w \in C_1$, while statement 2.5 together with the estimate $T_1^\circ(w) \leqslant \pi / 2$ shows that the first player cannot lessen the time to contact by leaving region $C_1$.

Note. The control $v_{2,0} = v^1$ realizes an escape from set $M_1$ in the region $C_2 \cap [\mu - \pi / 2 < 0]$. In fact, the difference $\mu - \pi / 2$ does not increase; therefore, the equation $q(w(\cdot, t), p) = 0$ cannot have at zero the value $p = 0$ along any admissible trajectory corresponding to the pair $u$, $v^1$.

3. In region $C_2$ the problem of constructing the minimax time becomes the problem of constructing the minimax time for the position to hit onto the set $M_2$ $[q(w, 0) = \mu - \pi / 2 - y = 0]$ in the presence of the phase constraint $w(\cdot, t) \in C_2$ (a constraint on the second player's actions). Let us reject the phase constraint for the present. Intuitively we feel also that the minimax is achieved for $u = 0$. Therefore, in the set $C_3$ $[\mu - \pi / 2 - y < 0]$ we seek the "slow-action" of $T_2$ on set $M_2$ and the control $v_2$ corresponding to it.

Within the framework of the "auxiliary" problem we form in the fixed system $(x_1, y_1)$ two three-dimensional vectors $g_x = \partial T_2 / \partial x_1$ and $g_y = \partial T_2 / \partial y_1$ and, after the operation $\max_v$, obtain the "fundamental equation" [1]

$$(g_x y_1) - (g_y x_1) + | g_y | + 1 = 0$$

with the "termination conditions" [1]

$$g_x^{(3)} = 0, \quad g_y^{(3)} = \lambda y_1 / y, \quad \lambda > 0 \quad w \in M_3 = M_2 \cap [y > 0, -y + x y_\alpha > 0]$$

The termination conditions show that the equalities

$$\begin{aligned} v_2 &= y_1 / y, \quad T_2 \leqslant \pi / 2, \\ v_2 &= -y_1 / y, \quad \pi > T_2 > \pi / 2, \end{aligned} \qquad y_1 / y \in M_3$$

are valid along the "characteristics" $g_x^{\boldsymbol{\cdot}} = g_y$, $g_y^{\boldsymbol{\cdot}} = -g_x$, i.e. control $v_2$ preserves along the characteristics a constant value of $y_1 / y$ equal to its value on set $M_3$ for $T_2 \leqslant \pi / 2$, and changes this value to a contrary one for $T_2 \geqslant \pi / 2$. It is clear that the optimal trajectories of the auxiliary problem remain in the plane containing the vectors $x_1$, $y_1$.

We choose a fixed system of axes coinciding with the moving trihedron $j_\alpha$, $j_\beta$, $j_\gamma$, typical for some position $w \in C_3$, and we denote the components along these fixed axes by subscripts $x_{1,i}, y_{1,i}, v_{2,i}$ $(i = 1, 2, 3)$. The discussions preceding show that $v_{2,3} = 0$. To determine $T_2 \leqslant \pi / 2$, $v_2$ we use the functions

$$y_{1,1} = -(x - v_{2,1}) a + y_\alpha b, \qquad y_{1,2} = v_{2,2} a + y_\beta b \tag{3.1}$$

$$a = \sin t, \quad b = \cos t$$

The condition $v_2 = y_1(T_2) / | y(T_2) |$ allows us to seek $T_2 < \pi / 2$ as the smallest positive root of the equation

$$\xi (w, t) = \mu - \pi / 2 - a - A (w, t) = 0$$
$$A (w, t) = [(-xa + y_\beta b)^2 + y_\beta{}^2 b^2]^{1/2}$$

To determine $T_2 > \pi / 2$ the same reasonings allow us to obtain the equation

$$\eta (w, t) = \mu - \pi / 2 - 2 + a - A (w, t) = 0$$

**4.** Let us establish a number of important properties of the function

$$\zeta (w, t) = \begin{cases} \xi (w, t \in [0, \pi / 2]) \\ \eta (w, t \in (\pi / 2, \pi)) = \xi (w, t - \pi) - 2 \end{cases}$$

4.1. The function $\xi (w, t \in [-\pi, \pi])$ has no more than two isolated maxima with respect to variable $t$.

**Proof.** For $w \in [[x = 0] \cup [y_\alpha = 0] \cup [y_\beta = 0]]$ statement 4.1 can be verified by calculation. At the remaining positions the function $A (w, t \in [-\pi, \pi])$ does not have zeros and the points $t_{1,2} = \mp \pi / 2$ are not stationary points for the function $\xi (w, t)$. We introduce the notation

$$\partial \xi / \partial t = \xi' (w, t \neq \pm \pi / 2) = -b - b^2 m_1 A^{-1}$$
$$m_1 = m_1 (w, t) = m_2 (w, z = \operatorname{tg} t) = (-xz + y_\alpha) (-x - y_\alpha z) + y_\beta{}^2 z$$
$$n_1 (w, t) = n_2 (w, z) = (-xz + y_\alpha)^2 + y_\beta{}^2 - m_2{}^2 (w, z)$$

A stationary point of function $\xi$ is a zero of the function $n_1$. On the other hand, a zero $t_j \in [-\pi / 2, \pi / 2]$ of function $n_1$ is a zero of the function $\xi'$ for $m_1 < 0$, or (for $(m_1 > 0)$ either the point $t_j + \pi < \pi$ or the point $t_j - \pi > -\pi$ is a zero of function $\xi'$ From the periodicity conditions and from the equalities $\xi (w, -\pi) = \xi (w, 0) = \xi (w, \pi)$ it follows that the function $\xi'$ has an even number of zeros, i.e. $k \geqslant 2$, while the function $n_2$ — a polynomial in $z$ of degree no higher than the fourth — has no more than four zeros on the interval $t \in (-\pi / 2, \pi / 2)$. This means that $k = 2, 4$.

4.2. The function $\xi (w, t)$ admits of no more than one maximum on the interval $[0, \pi)$.

The case when $\xi (w, t)$ has a maximum at $t = 0$, is investigated simply. It is not difficult also to show that the maxima and minima are isolated at the remaining positions. An accumulation of all the isolated points on the interval $[0, \pi]$ is impossible because of the equality $\xi (w, -\pi) = \xi (w, 0)$. It reamins to consider the possibility that two maxima $t_2{}^\circ$, $t_4{}^\circ$, separated by a minimum $t_{3,0}$, lie on the interval $[0, \pi]$, while the minimum $t_{1,0} \in (-\pi, 0)$. Since there are no other isolated points, the function $\xi (w, t)$ increases for $t \in [0, t_2{}^\circ]$. A contradiction follows from the estimate

$$\xi (w, t_2{}^\circ) > \xi (w, 0) > \xi (w, t_2{}^\circ - \pi) = \xi (w, t_2{}^\circ) + 2a (t_2{}^\circ)$$

The function $\zeta (w, t)$ is defined for $t \in [0, \pi]$. We say that $\zeta (w, t)$ has a "maximum at $t = 0$ " if the function $\xi (w, t \in [-\pi, \pi])$ has such a maximum at $t = 0$ . We note that the function $\xi$ has unisolated maxima at the sole position $w [x = 1, y = 0]$ . They are isolated at the remaining positions. In accordance to what has been presented, the function $\zeta (w, t \in [0, \pi])$ admits of no more than two maximum points $\tau_1 < \tau_2 \in [0, \pi)$ with the values $\zeta_1$ and $\zeta_2$ of the function $\zeta$. By $C_M$ we denote the region where even one of the maxima exists, and we set $\zeta_3 = \zeta_1$ when it is unique and $\zeta_3 = \max [\zeta_1, \zeta_2]$ when there are two maxima.

4.3. The equality $\zeta_1 = \zeta_3$ and the bound $\zeta_1 > \zeta_2$ if a second maximum exists, are valid in the region $C_M \cap [\zeta_3 \geqslant \zeta(w, 0) \equiv \zeta_0]$.

Proof. Let $\zeta_3 = \zeta(w, \tau_3 \in [0, \pi/2])$, then $\tau_3$ is a maximum point of function $\xi$. Statement 4.3 is verified simply when $w \in [x = 1, y = 0]$. At the remaining positions $\tau_3$ is an isolated point. In addition, according to 4.2, the estimate

$$\zeta_3 > \xi(w, t \in [0, \pi), t \neq \tau_3) > \zeta(w, t \in (\pi/2, \pi)) = \xi(w, t \in (\pi/2, \pi)) - 2 + 2a$$

is valid. Let $\tau_3 \in (\pi/2, \pi)$. From the estimate

$$\xi(w, t \in (\pi/2, \pi)) > \zeta(w, t) = \xi(w, t) - 2 + 2a$$

it follows that function $\xi$ has a maximum for $t \in (\pi/2, \pi)$ and, therefore, function $\zeta$ has only one maximum $\zeta_1 = \zeta_3$.

5. Let us consider the region $C_4 [\xi_0 = \mu - \pi/2 - y \leqslant 0]$, namely, the closure of region $C_3$. For $w \in A_1 [C_4 \cap [\zeta_1 \geqslant 0]]$ there exists a first nonnegative zero $t_\zeta$ of function $\zeta$. In the region $A_1' [A_1 \cap [A_\zeta \equiv A(w, t_\zeta) > 0]]$ the necessary conditions in section 5 lead to the control

$$v_2 = A_\zeta h_\zeta [(-xa_\zeta + y_\alpha b_\zeta) j_\alpha + y_\beta b_\zeta j_\beta] \tag{5.1}$$

$$a_\zeta = \sin t_\zeta, \quad b_\zeta = \cos t_\zeta, \quad h_\zeta = \begin{cases} 1, & t_\zeta \in [0, \pi/2] \\ -1, & t_\zeta \in (\pi/2, \pi) \end{cases}$$

A simple investigation shows that $t_\zeta$ can be the first zero of function $\zeta$ and that the function $A(w, t)$ vanishes only when $\zeta(w, t_\zeta) = \zeta_1$. Furthermore, the estimate $s(w) = x/(x^2 + y_\alpha^2) \leqslant 1$ is valid at these positions. However, in the region $A_1'' [A_1 \cap [\zeta_1 = A_\zeta = 0]]$ the necessary conditions do not yield a univalent control $v_2$ but yield only inequalities. Any control $|v| = 1$, satisfying these inequalities, is acceptable by the necessary conditions. For example, we can set

$$v_2 = h_\zeta s j_\alpha + \sqrt{1 - s^2} j_\beta \tag{5.2}$$

In the region $A_2 [C_4 \cap [\zeta_0 < 0, \zeta_1 \geqslant \zeta_0]]$ we continue the control by formulas (5.1), (5.2), by setting $t_\zeta = \tau_1$. As we shall see from what follows, there are sufficient grounds for such a choice. In the region $A_3 [C_4 \setminus (A_1 \cap A_2)]$ we seek to increase $y$ and, therefore, we set

$$v_2 = (y_\alpha j_\alpha + y_\beta j_\beta)/y, \quad w \in A_3 \cap [y > 0] \tag{5.3}$$

$$v_2 = x_1/x, \quad w \in A_3 \cap [y = 0, \quad x > 0]$$

$$v_2 = \text{const}, \quad |v_2| = 1, \quad w \in A_3 \cap [y = x = 0]$$

We introduce into consideration the trajectories $w_2$ generated by the pair $u_2 = 0$, $v_2$ and denote by $\dot{\zeta}_{0(2)}$ the derivative of the function $\zeta_0 = \zeta(w, 0)$ along trajectories $w_2$. We introduce also the set

$$M_4 [\zeta_0 = 0 \cap [[\zeta'(w, 0) = -1 + xy_\alpha/y \leqslant 0, \quad y > 0] \cup [y = 0]]]$$

which is the "reverse side" of set $M_2$, i.e. that part of it which the second player can always evade for $u = u_2 = 0$. We note that a trajectory $w_2$ can realize a "slow-action" onto $M_2$ from region $C_3$ if it does not intersect set $M_4$ until the set $M_2 \setminus M_4 \equiv M_3$.

has been hit. Unfortunately, not all trajectories $w_2$ possess this property and among them there exist those which from the region $C_3$ fall inside the region $C_5$ $[\zeta_0 < 0]$ through the boundary of $M_4$, and again go into region $C_3$ arriving at $M_3$ by the instant $t = t_\zeta(w(0))$. From the subsequent analysis it becomes clear in what sense we can proceed to talk about the trajectories $w_2$ in region $C_5$ wherein $v_2$ is not defined.

Lemma. Trajectories $w_2$ can intersect set $M_4$ from the side of region $C_3$ only on the set

$$M_5\,[M_4 \cap [1 < y < 2, \quad y_\alpha < 0, \quad \zeta_1 > 0, \quad t_\zeta > \pi / 2]]$$

Proof. Let $w \in M_6\,[M_4 \cap [\zeta_1 \geqslant 0, \ 0 < t_\zeta \leqslant \pi / 2]]$. From the equation $\zeta(w, 0) = \zeta(w, t_\zeta)$ we find that $t_\zeta$ is the first zero of the function $\lambda_1(w, t) = y - a - A$. From the equation $(y-a)^2 - A^2 = 0$ it follows that $t_\zeta$ is a zero of the function

$$\lambda_2(w, \ t) = (y^2 - x^2 + 1)\, a + 2xy_\alpha b - 2y$$

It is not difficult to show that $t_\zeta$ is its first zero. From the estimate $0 < t_\zeta \leqslant \pi / 2$ follows the estimate $-1 + xy_\alpha / y = \lambda_1'(w, \ 0) < 0$. This means that $t_\zeta$ satisfies the estimate

$$\lambda_2'(w, \ t_\zeta) = (y^2 - x^2 + 1)a_\zeta - 2xy_\alpha b_\zeta \geqslant 0$$

We now note that by the construction of control $v_2$ not one of the trajectories $w_2$ can make the function $A_\zeta$ vanish more than once, and for $A_\zeta = 0 \to w_2 \in C_3$. We can easily convince ourselves by looking over the possibilities. From the estimate $A_\zeta > 0$ and the equation $A_\zeta = y - a_\zeta$ follows the equality

$$\dot\zeta_{0(2)} = (xy_\zeta - b_\zeta) / y\,(y - a_\zeta)$$

It is not difficult to verify that the estimate $\dot\zeta_{0(2)} \geqslant 0$ is consistent with the estimate $\lambda_2'(w, \ t_\zeta) \geqslant 0$ and with the equality $\lambda_1 = 0$ only on the set

$$M_7\,[M_6 \cap [a_\zeta = (y^2 - x^2 + 1) / y, \ b_\zeta = xy_\alpha / y]]$$

For $w \in M_7$ the trajectory $w_2$ lies wholly in $M_7$, while $v$ "arrives" at $M_7$ from the region $C_5$ $[\zeta_0 < 0]$. We shall talk about the trajectories $w_2$ in region $C_5$ in the sense of a continuation into the "past" by a time $\tau = -\pi$ of the trajectories of set $M_3$ with a switching of control $v_2$ to the opposite one at $\tau = -\pi / 2$.

On the set $M_8\,[M_4 \cap [\zeta_1 \geqslant 0, \ t_\zeta > \pi / 2]]$ we have the equalities and estimates

$$\lambda_1 \equiv y - 2 + a_\zeta - A_\zeta = 0$$

$$\lambda_1' = (a_\zeta A_\zeta)^{-1}\,(4\,(y-1)b_\zeta - xy_\alpha a_\zeta - a_\zeta b_\zeta\,(y-2))$$

$$\dot\zeta_{0(2)} = (b_\zeta y^2 + xy_\alpha\,(y-2))\,(y^2\,(y - 2 + a_\zeta))^{-1}$$

$$T_\zeta > \pi / 2 \Rightarrow \zeta(w, \ \pi / 2) < 0 \Leftrightarrow x > y - 1$$

Assuming to the contrary that $\dot\zeta_{0(2)} \geqslant 0$, we obtain the estimate

$$\lambda_1 < \lambda_3 \equiv y - 2 + a_\zeta - [(y-1)^2 a_\zeta^2 + y^2 b_\zeta^2 + 2a_\zeta b_\zeta^2 y / y - 2]^{1/2}$$

In the region $M_8 \cap [y \geqslant 2]$ we can easily verify the estimate $\lambda_3 < 0$. The contradiction with the equation $\lambda_1 = 0$ leads to the estimate

$$\dot\zeta_{0(2)}\,(w \in M_8 \cap [y \geqslant 2]) < 0$$

In the region $M_8 \cap [\zeta_1 = 0]$ the equality $\lambda_1' = 0$ implies, as a consequence, the relation

$$\dot{\zeta}_{0(2)}\ (w \in M_8 \cap [\zeta_1 = 0]) = 4\,(y-1) b_\zeta A_\zeta / a_\zeta < 0$$

In the region $M_8 \cap [y \leqslant 1]$ the function $\lambda_1$ has no zeros.

At the remaining positions of the set $M_4 \setminus M_5$ it is clear by the construction of $v_2$ that $\dot{\zeta}_{0(2)} < 0$. An exception is the set $M_4 \cap [x = 1,\ y = 0]$, i.e. the "fixed points" which the trajectories $w_2$ cannot hit from the side of region $C_3$. We note that in the proof we used the equation $\zeta_0 = \zeta\,(w,\ t_\zeta)$ and did not use the equation $\zeta_0 = 0$. This means that the trajectory $w_2 \in A_2$ can once again pass into $A_3$ from the boundary $[\zeta_0 = \zeta_1 < 0,\ t_\zeta > 0]$ of regions $A_2$ and $A_3$ only for $t_\zeta = 0$. Sliding states are impossible on this boundary whereon $v_2$ is discontinuous.

Let us fix $y$ on set $M_4$ and increase $x$ from the value $x_1 = y - 1$. The function $\dot{\zeta}_{0(2)}$ will change from the value $(\dot{\zeta}_{0(2)})_1 = x_1 y_\alpha (2-y) > 0$. With the change in $x$ the function $\zeta_1$ (the maximum) necessarily changes sign at $x = x_2$. According to the lemma the estimate $\dot{\zeta}_{0(2)} < 0$ is valid for $x = x_2$. This means that the equation $\dot{\zeta}_{0(2)}\,(x_3,\ y_\alpha,\ y) = 0$. is valid for some $x_3 = x_3\,(y_\alpha, y)$.

For $x = x_3$ it is not difficult to determine the equality and the estimate

$$\partial \dot{\zeta}_{0(2)} / \partial x = \zeta'\,(w,\ t_\zeta)^{-1}\, c\,(w)$$

$$c\,(w) = 4\,(y-1)\,A_\zeta - y^2 a_\zeta^3\,(y^2 a_\zeta / (2-y) + y_\alpha) < 0$$

which establish the uniqueness and continuity of the curve $x = x_3\,(y_\alpha,\ y)$ separating the regions $N_1\ [M_4 \cap [\dot{\zeta}_{0(2)} < 0]]$ and $N_2\ [M_4 \cap [\dot{\zeta}_{0(2)} > 0]]$ for any value $1 - y < 2$. In region $N_2$ we form a new control $v_2$ and instead of a realization by formula (5.1) we set

$$v_2\,(w \in N_1) = v_{2,\alpha}\, j_\alpha + \sqrt{1 - v_{2,\alpha}^2}\, j_\beta \tag{5.4}$$

$$v_{2,\alpha} = x y_\alpha / y^2 + (x^2 y_\alpha^2 / y^4 - y_\alpha^2\,(1 + x^2) / y^2 + 1)^{1/2}$$

This control maximizes the derivative $t_\zeta^{\boldsymbol{\cdot}}\,(w,\ u_2,\ v)$ on the set of controls $v$ preserving $(\dot{\zeta}_0\,(w,\ u_2,\ v) = 0)$ the value $\zeta_0$. In the region $C_2$ also we form the control $v_3\,(w)$ by the formulas

$$v_3\,(w) = v_2\,(w), \quad w \in C_2 \cap [r_1\,(w) \leqslant \zeta_1] \equiv F_1 \tag{5.5}$$

At those positions where $r_1\,(w) > \zeta_1$, or where $\zeta_1$ does not exist, we set

$$v_3\,(w) = v_1\,(w), \qquad w \in C_2 \setminus F_1 \equiv F_2 \tag{5.6}$$

6. The control $v_2$ realizes the time $T_2\,(w) = t_\zeta\,(w)$ for all trajectories $w_2$ arriving at $M_3$ from region $C_3$ and by-passing set $N_1$. Trajectories $w_2$, hitting onto $N_1$, arrive at $M_3$ by the instant $T_2\,(w) = t_1\,(w) + t_2\,(w) + \pi / 2$. Here $t_1\,(w)$ is the time of motion up to set $N_1$, $t_2\,(w)$ is the time of sliding on set $N_1$ from which the trajectory leaves for $y - 1 - x = 0,\ y_\alpha \leqslant 0$, i.e. at the instant the equality $t_\zeta\,(w) = \pi / 2$. is realized. We can show that the trajectory does not intersect the curve $x = x_3$.

The region occupied by trajectories of the first type is denoted by $W_2^\circ$ (max) ; trajectories of the second type occupy the region $W_2^\circ$ (sup). The rest of the trajectories $w_2$ starting in $C_3$ occupy the region $W_{0,2}$.

Theorem 6.1. The equality $T_2^\circ\,[u_2^\circ = 0,\ \ v_2^\circ = v_2] = T_2^\circ = T_2\,(w)$ realizes in region $W_2^\circ$ (max) . In region $W_2^\circ$ (sup) there exists a sequence of controls $v_{2,j}$ such that

$$T_2[u,\ v_{2,j}] \leqslant T_2[u_2^\circ = 0,\ v_{2,j}] \to T_2(w) \quad \text{as}\ v_{2,j} \to v_2$$

In the region $W_{0,2}$ the control $v_2 = v_{0,2}$ causes the trajectory to evade set $M_2$.

We present a short plan of proof for Theorem 6.1. The estimate

$$T_2^{\cdot}[w,\ u_2,\ v] \leqslant T_{2(2)}^{*} \leqslant T_2^{\cdot}\ [w,\ u \neq 0,\ v_2]$$

is valid in the regions $W_2^\circ$ (max) and $W_2^\circ$ (sup). The proof of this estimate (for finite controls $u$) can be carried out in the region $W_2^\circ(\max) \cap [\zeta_1 > 0]$ by the implicit function theorem. In the region $W_2^\circ(\max) \cap [\zeta_1 = 0]$ the second player's errors lessen $T_2(w)$ with an "infinitely great" rate, while the first player's errors translate the position into region $W_{2,0}$.

The proof in the region $W_2^\circ$ (sup) is complicated; however, it can be carried out even so. It is clear that in the region $W_2^\circ$ (sup) the second player can pass to control (5.4) when $y = \mu - \pi / 2 - \varepsilon$ (where $\varepsilon$ is a small quantity) and can obtain a time as close as desired to $T_2(w)$, while the first player cannot lessen even this nonoptimal time. The first player's impulse actions also can only increase the time $T_2$ or translate the position into region $W_{0,2}$. In region $W_{0,2}$ the control $v_2$ is constructed so that the function $r_2(w) = \max \zeta(w,\ t \in [0,\ \pi))$ cannot be increased along trajectory $w_2$ and has a negative initial value. In fact, in the lemma it was shown that $\zeta_0$ does not increase when $\zeta_0 = \zeta_1$ or at the positions where $\zeta_0$ is a unique maximum point. On the other hand, when $\zeta_0 = \zeta_1$ (or when $\zeta_0 < \zeta_1$) the maximum $\zeta_1$ is preserved, since the equality $\dot{\zeta}_{1(2)} = 0$ is easily verified. The first player cannot increase the function $r_2(w)$ by a finite control or by an impulse and the position remains in region $W_{0,2}$ the whole time. These arguments together with the continuity of the function $t_\zeta$ (which follows from 4.3) and of function $T_2(w)$ yield a sufficient basis for the proof of Theorem 6.1.

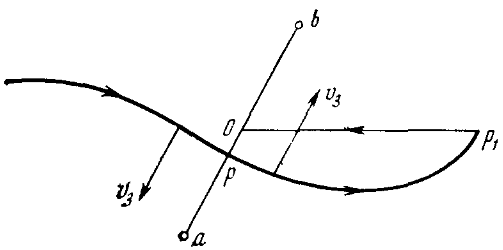


Fig. 1

We now return to the original problem and to the curves $w_3$ generated by the pair $u_3 = 0$, $v = v_3 = v_2$. Their study introduces the important question of the sign of the derivative $\dot{r}_{1(3)}$ of the function $r_1(w)$ along the curves $w_3$ on the set $C_6[r_1 = \zeta(w,\ t_\zeta)]$. This question is difficult because the functions $r_1$ and $t_\zeta$ are not specified explicitly. We have succeeded in proving the estimate $\dot{r}_{1(3)} < 0$ in the region

$$C_7[C_6 \cap [[y_\beta = 0] \cup [y_\alpha = 0] \cup [t_\zeta = \pi / 2 \pm \varepsilon]]]$$

The notation $t_\zeta = \pi / 2 \pm \varepsilon$ was used because control $v_3$ is discontinuous for $t_\zeta = \pi / 2$, but the estimate $\dot{r}_{1(3)} < 0$ was established for both limits of $v_3(w)$. If the estimate $r_{1(3)} \leqslant 0$ is valid for $w \in C_6$, then it can be shown that the equalities

$$T_1(w \in A_1 \cap C_3) = t_\zeta + \pi / 2, \quad W_{0,1} = (A_2 \cup A_3) \cap C_3$$
$$v_3 = v_{0,1}$$

are valid. However, if the estimate $\dot{r}_{1(3)} \leqslant 0$ is violated, then additional investigation is required. In this case the pair $u_{1,0} = 0$, $v_{1,0} = v_3$ is optimal only in some region which can be constructed by continuing the trajectories $w_3$ into the past (coinciding in this case with trajectories $w_2$) up to intersection with the surface $r_1(w) = 0$.

Figure 1 shows a typical trajectory $w_3$. At the start the position is moved along an ellipse with center at point $a$ and the control $v_2$ has a constant direction up to the switching point $p$. After the switching at point $p$ motion takes place along an ellipse with center at point $b$. The lengths of the segments $(a, 0)$, $(0, b)$ equal unity. At the point $p_1$ $(\mu - \pi / 2 - y = 0)$, lying on set $M_3$, the first player turns off the velocity by impulse and the position is moved to "hard" contact during a time $\pi / 2$.

We fix a certain small number $\varepsilon_1 > 0$ and among the trajectories $w_2$ we isolate a family $w_{2,\varepsilon,1}$ by the following test. Along any trajectory $w_{2,\varepsilon,1}$ of the family indicated, from the estimate $p_1(\dot{w}_{2,\varepsilon,1}) < 0$ follows the estimate $r_1(w_{2,\varepsilon,1}) \leqslant -\varepsilon_1$, while from the estimate $r_1(w_{2,\varepsilon,1}) > -\varepsilon_1$ follows the equality $p_1(w_{2,\varepsilon,1}) = 0$. Suppose that the trajectories $w_{2,\varepsilon,1}$ occupy a region $W_{\varepsilon,1}$. We state the final result.

Theorem 6.2. The controls $u_1^\circ = u_3 = 0$ and $v_1^\circ = v_2$ realize, in the region $W_{\varepsilon,1} \cap W_2^\circ(\max)$ the time $T_1 = t_\zeta + \pi / 2$ and the second player cannot increase this time. This time cannot be lessened by the first player by any pair $u$, $v_2$ preserving the inclusion $w \in W_{\varepsilon,1} \cap W_2^\circ(\max)$. If the inclusion indicated is not violated until $M_2$ is hit, then the motion passes into region $C_1$ through the boundary $T_1 = \pi / 2$ $(t_\zeta = 0)$.

## REFERENCES

1. Isaacs, R. P., Differential Games. New York, John Wiley and Sons, Inc., 1965.
2. Krasovskii, N. N., Game Problems on the Contact of Motions. Moscow, "Nauka", 1970.
3. Pozharitskii, G. K., Game problem of the hard contact of two points with an impulse thrust in a linear central field. PMM Vol. 36, №6, 1972.
4. Pozharitskii, G. K., Impulse tracking of a point with bounded thrust. PMM Vol. 37, №2, 1973.

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# ON OPTIMUM SELECTION OF NOISE INTERVALS IN DIFFERENTIAL GAMES OF ENCOUNTER

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We examine differential games of encounter in which the minimizing player observes the game's position on a subset $Q$ of the motion interval $[t^\circ, T]$. The subset $Q$ is formed by the second player during the motion, i.e. he switches on a noise eliminating observation. We pose the problem of optimal noise distribution and solve four examples. A general setting of similar problems was given in [1]. Related problems were examined, for example, in [2, 3].